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Capacitors can radiate

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Indice

- 1 Abstract
- 2 Introduction
- 3 Lumped model
- 4 Radiation
- 5 Hybrid model
- 6 Adding R
- 7 Adding L
- 8 Conclusions
- 9 Appendice A
- 10 References
- 11 Note

Abstract

Capacitor paradox, **two-capacitor problem**, **two capacitor with radiation**, call it as you like, anyway this parallel is anything but simple!**!!!**

I state that my own is just an attempt to summarize the work of Timothy B. Boykin, Dennis Hite, and Nagendra Singh in [2] and TCChoy in [4].

Where possible, I've tried to use the same symbolism adopted by those references, to which we refer for a more extensive and complete treatment.

Warning The reading of this article is strictly contraindicated in individuals who have strong allergic reactions to mathematics ... and electrotechnics!

Introduction

The problem that we will study in this article is that of the energy balance in a simple parallel between two capacitors.

Considering the insertion of two parallel capacitors, charged at different voltages, one can verify that, there is a difference between the total energy stored in the system before switch on and the total final one.

Assuming for simplicity two equal capacitors, and only one capacitor charged, the total energy of the initial and final system will be

$$W_{initial}=\frac{1}{2}\frac{Q^{2}}{C}\,\,\,\,\,e\,\,\,\,\,W_{final}=2\cdot \left(\frac{1}{8}\frac{Q^{2}}{C} \right)=\frac{1}{4}\frac{Q^{2}}{C}\,\,\,\,\,with\,\,\,\Delta W=-\frac{1}{4}\frac{Q^{2}}{C}$$

witha loss exactly equal to half the initial one .

The problem can be solved if we consider a simplified lumped model with the loop resistance R (and / or inductance L), always present in reality, but we want to know if the result is acceptable for R going to zero.

Lumped model

Consider as initial model for the study of the system, a lumped circuit model comprising two capacitors and one resistor R in series. We will show that, if accepted this circuit representation, the missing energy equals that dissipated by Joule effect by the resistance R.

fig. 1

fig. 1

Power on **R** will be

$$W_{R}= \int\limits_{0}^{\infty }{v_{R}(t)}\cdot i_{R}(t)\,dt$$

and then

$$W_{R}=\int\limits_{0}^{\infty }{\left[v_{1}(t)-v_{2}(t) \right]}\cdot i(t)\,dt=\int\limits_{0}^{\infty }{v_{1}(t)}\cdot i(t)\,dt-\int\limits_{0}^{\infty }{v_{2}(t)}\cdot i(t)\,dt$$

recalling that $i(t)dt$ will be equal to the decrease of charge **-dq** in the first capacitor and to the increase of charge **+dq** in the second one,

$$W_{R}=\frac{1}{C}\int\limits_{Q}^{\frac{Q}{2}}{q_{1}(t)}\cdot (-dq)-\frac{1}{C}\int\limits_{0}^{\frac{Q}{2}}{q_{2}(t)}\cdot dt=\left[-\frac{1}{2}\frac{q^{2}(t)}{C} \right]_{Q}^{\frac{Q}{2}}-\left[\frac{1}{2}\frac{q^{2}(t)}{C} \right]_{0}^{\frac{Q}{2}}=\frac{1}{4}\frac{Q^{2}}{C}=\Delta W$$

equal to the missing energy ΔW.

As the result is independent of R value, the problem seems solved.

A critical analysis of the results, however, makes us realize that, with serie resistance close to zero, the high currents and low time constants, bring into serious doubt the validity of the model , without doubt a part of the radiated power will be lost !

A more complete classical analysis is treated in Rif. [8], [1] e [5].

Radiation

- Initial Considerations

The choice at this point, seems forced: describe the system in its geometry, leaving the lumped model for a numerical analysis of the phenomenon, hard but feasible task.

Wanting to remain in a concentrated parameters model, and still use symbolic calculation, we'll assume that power is a function of time but not of spatial coordinates and make use of a model that we can define "hybrid".

The geometry will be treated for simplicity as that of a circular loop of radius b, symmetrically cutted by the two capacitors.

fig. 2

fig. 2

To be able to accept the "hybrid model", we should check Abraham's condition, that we should assume a time constant much greater than propagation time , said b the radius of the circuit and c the speed of light, you should assume that

$$\tau \gg \frac{2\pi b}{c}.$$

Estimated as in [2] a radius '*b*= 5 cm', we will have a validity of the results obtained up to a τ of the order of nanosecond.

A further simplification will be to not consider the electric dipole radiation, but only that of "magnetic dipole" believing it to be predominant; in [4], however, Choy use this option to complete Boykin's analysis [2].

In the lumped model, the power lost in electromagnetic radiation is taken into account by introducing in the circuit a virtual resistance Rrad (X element), which will lead to a nonlinear differential equation .

Its resolution will provide the Vc (t) , equal to the difference between the voltages on the two capacitors.

- **Retarded vector potential**

The representation of the electromagnetic field through the introduction of a scalar electric potential and magnetic vector potential, in no steady condition should be extended to a dependency on propagation time, then we will talk of retarded potential [6]. In our problem, due to initial assumptions, we will consider only the magnetic potential, defined with reference to fig.3

fig. 3

fig. 3

from

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu _{0}}{4\pi }\int\limits_{V}{\frac{\mathbf{J}({\mathbf{r}}',t{}_{r})}{\gamma}}\,dv\qquad\qquad\qquad\qquad(1)$$

where

$$\gamma =\left| \mathbf{r}-{\mathbf{r}}' \right|\,\,\,\,\,\,\,\,\,\,t_{r}=t-\frac{\gamma }{c}$$

that for our configuration, due to z symmetry

fig. 4

fig. 4

with $dv=s\cdot b\cdot d{\varphi }'$, where we can write

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu _{0}}{4\pi }\int_{0}^{2\pi }{\frac{I(t-\frac{\gamma }{c})}{\gamma }}\cdot\mathbf{e}_{\varphi }\,b\,d{\varphi }'$$

in which the current $I=J\cdot s$is oriented as the azimuthal versor $\mathbf{e}_{\varphi }$.

Through the retarded vector potential , we can derive both the electric field and the magnetic induction

$$\mathbf{E}=-\nabla \mathbf{\Phi} -\frac{\partial \mathbf{A}}{\partial t}\,\simeq \,-\frac{\partial \mathbf{A}}{\partial t}\,\,\,\,\,\,\,\,\,\,e\,\,\,\,\,\,\,\,\,\,\mathbf{B}=\mathbf{\nabla} \times \mathbf{A}$$

thanks to the negligible contribution of the gradient of electrical potential.

We are now able to derive the *'Poynting vector'*

$$\mathbf{S}(\mathbf{r},t)=\frac{1}{\mu _{0}}\mathbf{E}(\mathbf{r},t)\times \mathbf{B}(\mathbf{r},t)$$

its flux through a spherical surface of radius **r** and center **o**,

$$P_{rad}=\int_{0}^{2\pi }{\int_{0}^{\pi }{\mathbf{S}\cdot \mathbf{e_{r}}}}\,r^{2}\sin (\theta \,)\,d\theta \,d\varphi$$

will give us the power radiated by the loop.

$$P_{rad}=\frac{\pi b^{4}}{6\varepsilon _{0}c^{5}}\left[\ddot{I}\left(t-\frac{r}{c} \right) \right]^{2}\qquad\qquad\qquad\qquad(2)$$

Hybrid model

- Radiation Resistance

The equation (2) allows us to define the value R_{rad} of the "equivalent resistance" of the non-linear element **X** which represents the power lost by electromagnetic radiation

fig. 5

fig. 5

The voltage drop on this element will be

$$V_{X}=\frac{P_{rad}}{I}=K\cdot \frac{{\ddot{I}}}{I} \,\, \,\,\,with\,\,\,\,\,K=\frac{\pi b^{4}}{6\varepsilon _{0}c^{5}}$$

Assuming a limit ohmic resistance **R** equal to zero and if we define

$$V_{c}=V_{2}-V_{1}\,\,\,\,\,\,\,and\,\,\,\,\,\,C_{s}=\frac{C}{2}\,\,\,\,\,\,\,\,,$$ series of the capacitors

we can write the following differential equation (see note 1)

$$\left(\frac{d^{3}V_{c}}{dt^{3}} \right)^{2}+\frac{1}{KC_{s}}\,\frac{dV_{c}}{dt}V_{c}=0$$

and trying a solution like

$$\,\,V_{c}=Ae^{st}.$$

From the corresponding characteristic polynomial

$$s=\left(\frac{1}{KC_{s}} \right)^{\frac{1}{5}}e^{j\left(2n+1 \right)\frac{\pi }{5}}\,\,\,\,\,\,\,\,\,(n=0,1,2,3,4)$$

among which, the only real solution will be the one corrisponding to $n = 2$,

$$V_{c}(t)=V_{0}e^{-\alpha t}\,\,\,\,\,\,\,con\,\,\,\,\,V_{0}=\left[V_{2}(0-)-V_{1}(0-) \right]\,\,\,\,\,\,\,and\,\,\,\,\,\,\alpha =\left(\frac{1}{KC_{s}} \right)^{\frac{1}{5}}$$

and the current through a simple differentiation

$$I(t)=C_{s}\frac{dV_{c}(t)}{dt}.$$

- Final Result

We are now able to calculate the integral of Poynting to find the total energy radiated

$$W_{rad}=\int_{0}^{\infty }{P_{rad}\,dt}=\int_{r/c}^{\infty }{P_{rad}\,dt=K\int_{r/c}^{\infty }{\alpha ^{5}C_{s}^{2}V_{0}^{2}}}\alpha \,e^{-2\alpha \left(t-\frac{r}{c} \right)}\,\,dt=$$
$$=\frac{1}{2}C_{s}V_{0}^{2}=\frac{1}{4}CV_{0}^{2}$$

Exactly equal to energy lost between the two initial and final states of the system.

(NB integration may start from time R, due to the propagation delay time, by which the switch is perceived at a distance r.)

The result is really interesting, even without ohmic resistance, the circuit has evolved exponentially due to the presence of an equivalent radiation resistance**".**

Adding R

If we consider the ohmic resistance, the differential equation becomes more complicated

$$\left(\frac{d^{3}V_{c}}{dt^{3}} \right)^{2}+\frac{R}{K}\left(\frac{dV_{c}}{dt} \right)^{2}+\frac{1}{KC_{s}}\,\frac{dV_{c}}{dt}V_{c}=0$$

that would generally be solved only by a numeric way

we will not give here the detailed complete solutions, given in Appendix A,but only the results related to a current loop of radius **b=5 cm** and an equivalent series capacitance equal to **100** μF.

The following chart shows, depending on the ohmic loop resistance'*R* ***, the ratio of radiation resistance to real*** **Joule** resistance , and also the decay constant related to **Vc(t)**.

fig. 6

fig. 6

As we can see, for **R** greater than some tenths of milliohms, the time constant is still approximately equal to **RC** and the ratio between radiated power and Joule power dissipated in **R** is less than **1/1000**.

$$R>0.3\,m\Omega \,\,\,\,\,\to \,\,\,\,\,\,\tau \approx RC\,\,\,\,\,e\,\,\,\,\,\frac{P_{rad}}{P}=\frac{R_{rad}}{R}<\frac{1}{1000}$$

[From fig.6, for R=0.3 milliohm, we read -0.033 for s, so $\tau \approx 30\,\,ns$ (point Q) and 0.8/1000 for power ratio (point P)].

We can divide the graph into three main zones (fig. 7):

- **A**,in which the influence of radiation resistance is negligible,and where we see how, s(R) functions and power ratio are well approximated by the simple 1/RC, the former, and by the **r** line the latter, because in A zone **A**

$$R_{rad}=K\,s^{4},\,\,\,\,\,\,\,\,\left| s \right|=\frac{1}{RC},\,\,\,\,\,\,\,\,\,\frac{P_{rad}}{P}=\frac{R_{rad}}{R}\propto \frac{1}{R^{5}}$$

fig. 7

fig. 7

- **B**, characterized by a gradual transition between the weight of the two powers, the point of equality in our case will be for R equal to 40 microohm,
- **C**, where the radiation resistance assumes a predominant role; time constant saturate near **7 ns** (1 / 0.146) allows us to validate the initial assumptions on the time allowed for propagation in our hybrid model.

Adding L

To complete the analysis at this point it is necessary to take into consideration the coefficient of autoinduction L, certainly not zero for a real loop ,

fig. 8

fig. 8

lead to a more complex differential equation

$$\left(\frac{d^{3}V_{c}}{dt^{3}} \right)^{2}+\frac{L}{K}\left(\frac{d^{2}V_{c}}{dt^{2}} \right)\left(\frac{dV_{c}}{dt} \right)+\frac{R}{K}\left(\frac{dV_{c}}{dt} \right)^{2}+\frac{1}{KC_{s}}\,\left(\frac{dV_{c}}{dt} \right)V_{c}=0$$

The Vc analysis can be obtained only through numeric solution and is restricted in [2], to particular values of circuit parameters R, L and C.

To restrict the number of possible choices will be useful to evaluate the order of magnitude of the resistance and inductance associated with the geometric model.

Assuming a copper conductor with radius **a** equal to 0.5 mm,

$$R=\rho \cdot \frac{2\pi b}{\pi a^{2}}=0.0176\cdot \frac{10\cdot 10^{-2}}{0.5^{2}}=\frac{17,6\cdot 10^{-4}}{25\cdot 10^{-2}}=7\,\,m\Omega$$

This parameter should be considered only indicative, as the inverse square dependence by **a** let for R large range of values.

For the inductance associated with the circular geometry use

$$L=4\pi b\left(\ln \frac{8b}{a}-\frac{7}{4} \right)10^{-7}=0.3\,\mu H$$

in this case, the inverse logarithmic dependence of *'a'* will lead to less sensitivity to the diameter of the wire and reduce the possibility of variation of L mainly to a reduction in the development of conductive connection between the two capacitors.

With these values, the circuit will always under damped and the radiation resistance, at the resonant frequency , will be generally negligible compared to the ohmic one, also reducing the capacity to 1 nanofarad .

$$R_{rad}\approx Ks^{4}\approx K\frac{1}{\left(\sqrt{LC} \right)^{4}}=1.7\cdot 10^{-6}\,\,\,\Omega$$

$$\frac{R_{rad}}{R}\approx \frac{1}{4000}$$

In [2], while underlining the limit condition, it is supposed to reduce to ten microohm R and L to a nanohenry to still be able to achieve equality between ohmic resistance and radiation

In my humble opinion impracticable conditions, since such a reduction of resistance would have to take into account the series resistance of the capacitor (previously neglected), and the reduction in the self induction to few nanohenry would also be impractical, as I have wrote, without any connections reduction.

For further information please refer to original Article [**2**] and the new Choy's document [**4**] which addresses the topic, generalizing both radiative types.

Conclusions

As we have seen the simplified model was not initially able to properly represent the circuit in the case of particularly high currents and low decay constants. The consideration of the radiated power and its representation through a virtual resistance in hybrid model, has allowed us to demonstrate how, for values of ohmic resistance close to zero, there is a gradual shift from predominantly dissipative effect of a predominantly radiative effect.

Practical considerations have shown, for a real circuit configurations, a radiative correction of the phenomenon, that only in extreme cases may represent a significant part of the power lost from the system.

Appendice A

For a numeric solution of the

$$\left(\frac{d^{3}V_{c}}{dt^{3}} \right)^{2}+\frac{L}{K}\left(\frac{d^{2}V_{c}}{dt^{2}} \right)\left(\frac{dV_{c}}{dt} \right)+\frac{R}{K}\left(\frac{dV_{c}}{dt} \right)^{2}+\frac{1}{KC_{s}}\,\frac{dV_{c}}{dt}V_{c}=0$$

$$s^{5}+\frac{L}{K}s^{2}+\frac{R}{K}s+\frac{1}{KC_{s}}=0$$

The equation will be characterized, as appropriate, by setting the values of circuit parameters.

For the numerical calculation of the roots, we can use wxMaxima, enter the values for L, C, R, K, and define the function f(x), with "allroots" you can obtain the zeros of the function fig. 10

fig. 10

in the example we have used a LC loop with R=0.

Wanting to derive a complete graphical representation of changes in both the relationship between radiation resistance Rrad and ohmic resistance R, is the function s(R), it is more convenient to use Scilab.

Just a script to calculate and draw the curves

```
  //  ------ Copyright ---  Renzo Del Fabbro  2009  -----

clear
format('e',21);
p=%pi;j=%i;c=3e8;b=5e-2;eps=8.8541878176e-12;
r=1e-8;L=0;C=1e-4;K=p*b^4/(6*eps*c^5);
for i=1 : 5
  for j=1 : 10
   k=10*(i-1)+j;
   R=r*(10^i)*j;
   coef=[1/(K*C),R/K,L/K,0,0,1];
   fs=poly(coef,'x','coeff');
   radici=roots(fs);
//---------- store the results------------------
   VR(k,1)=radici(1);
//---- just a trick to pickup some real solutions
      if k == 12| k == 18   then VR(k,1)=radici(3);
       end
  VR(k,2)=R;
  VR(k,3)=(1/(abs(VR(k,1))*C)-R)/R ;
 end
end
//----- Draw curves -------------
 xbasc()
 xgrid
 x=VR(:,2);
 y=VR(:,3);
 z=VR(:,1)*1e-9;
 subplot(2,1,1)
 xtitle("Resistenza di radiazione/ Resistenza Joule")
 plot2d(x, [y y] , [5 -1],logflag='ll' )
 subplot(2,1,2)
```

```
xgrid
xtitle("Inverso costante di tempo s (ns-1)")
plot2d(x , [z z] ,[3 -1], logflag='ln')
xselect()
```

You get the following two graphs

fig. 11

fig. 11

equal to those of Figure 5 in Boykin's article [2].

You notice the 50 values calculated for the real root of the associated equation, the conjugated imaginary pairs solutions are not acceptable, as shown in [2].

References

[1] K. Mita and M. Boufaida, "Ideal capacitor circuits and energy conservation", Am. J.Phys. 67, 737 (1999).

[2] Timothy B. Boykin, Dennis Hite, and Nagendra Singh, "The two-capacitor problem with radiation", Am. J. Phys. 70, 415 (2002).

[3] Kirk T. McDonald, "A Capacitor Paradox", J. Henry Labs. Princeton University (2002).

[4] T.C. Choy, "Capacitors can radiate - some consequences of the two-capacitor problem with radiation", Am. J.Phys. 72, 662 (2004).

[5] R.A. Powell, "Two-capacitor problem", Am. J. Phys. 47, 460 (1979).

[6] V. Degli Esposti, "I potenziali elettrico e magnetico", dispense del corso di Propagazione.

[7] David J. Griffiths, Introduction to Electrodynamics, pag.451- Prentice Hall, 1999.

[8] A. Pramanik ,Electromagnetism Problems With Solutions, pag.100- PHI Learning 2008

Note

1) In [2] equation (12) page. 416 should be corrected by replacing the second derivative \ddot{V}^{2} with the third derivative, again squared. Mr. Boykin can be contacted at **boykin@eng.uah.edu** .

2) The paper [4] previously linked, has been replaced with a newer version, kindly provided by Prof. Tuck Choy - '*tuckchoy@ieee.org* ***which is available through*** **'Skype** for a discussion on the subject (an unmissable opportunity for those that find these four lines incomplete).

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